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Relaxation of nonlinear oscillations in BCS superconductivity

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Abstract

The diagonal case of the $sl(2)$ Richardson–Gaudin quantum pairing model is known to be solvable as an Abel–Jacobi inversion problem. The effect of random time-dependent perturbations of the single-particle spectrum on the exact solution is an open question of considerable physical relevance. Weak perturbations introduce a new, slow time scale, while preserving the nonlinear character of the dynamics. In this paper, such perturbations are considered. It is shown that the long-time asymptotics can be obtained by a deformation of the original integrable system, equivalent to phase averaging over the fast time scale.

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1. Introduction

The integrable system described by the pairing Hamiltonian introduced by Richardson and Sherman [1–6] in the context of nuclear physics, the diagonal case of which is known to be solvable as an Abel–Jacobi problem [7–15], has received revived interest in recent years, after being applied to metallic superconducting grains [16] and cold fermionic systems [15, 17]. The model is intimately related [18–20] to a class of integrable systems generally referred to as Gaudin magnets [7]. These systems have been studied both at quantum and classical level [21–30], in the elliptic case as well as trigonometric and rational degenerations, using various methods from integrable vertex models to singular limits of Chern–Simons theory.

In [17], an interesting regime of the pairing problem was considered, which may be relevant to recent experiments with cold fermionic gases exhibiting the paired BCS state [31–38]. It was shown that for such systems, the time scales of the order parameter $\tau_\Delta \sim |\Delta|^{-1}$, and the quasiparticle energy relaxation time τ_ϵ are both much larger than typical time for switching on the pairing interaction τ_0, essentially given by the variation of external parameters,

such as detuning from the Feshbach resonance. It was argued that in this regime, for times $t \ll \tau_\epsilon$, the dynamics of the system is given by nonlinear, non-dissipative equations describing the coherent BCS fluctuations for the system out of equilibrium. In this limit, the system is integrable, and features non-perturbative behaviour, such as soliton-type solutions.

In the mean-field limit, such non-trivial solutions describing the collective mode of the Anderson spins [39] were derived in [17], for a two-level effective system. This work was generalized [15] in algebro-geometrical terms.

In [40–42], the long-time behaviour of the solution has been considered, under various conditions. An issue not addressed so far is the relaxation of the nonlinear oscillatory solution induced by perturbations of the spectral curve, physically justified by coupling to the environment. Several possible kinds of perturbations may be considered, which may lead to different types of relaxation.

In this paper, we consider the effect of weak, uncorrelated fluctuations $\delta\epsilon_i(t)$ of the single-particle energy levels, such that the variance $\sigma_\delta = \overline{(\delta\epsilon_i)^2} \ll |\Delta|^2$. These fluctuations introduce a long time scale $\tau_\delta = O\left(\sigma_\delta^{-1/2}\right) \gg \tau_\Delta$. We identify the effect of these fluctuations with phase averaging over the (original) fast time scale of the unperturbed solution. This amounts to slow deformations of the Liouville tori, and can be described by hydrodynamic-type equations in phase space [43, 44]. These equations describe the evolution of the moduli for the complex curve of the system. It is shown that, in the generic case, the dependence of the gap parameter on the slow time is of the form $\mathrm{e}^{\tau_\delta/t}$.

The paper is organized as follows: in section 2, the Richardson–Gaudin model and its semiclassical limit are reviewed. Section 3 describes the phase averaging method in the case of hyper-elliptic spectral curves. In section 4, this method is applied to the nonequilibrium BCS problem.

2. The Richardson–Gaudin model

2.1. The quantum pairing Hamiltonian

Following [21], we briefly review the Richardson pairing model. It describes a system of n fermions characterized by a set of independent one-particle states of energies ϵ_l, where the label l takes values from a set Λ. The labels may refer, for instance, to orbital angular momentum eigenstates. Each state l has a total degeneracy d_l, and the states within the subspace corresponding to l are further labelled by an internal quantum number s. For instance, if the quantum number l labels orbital momentum eigenstates, then $d_l = 2l + 1$ and $s = -l, \ldots, l$. However, the internal degrees of freedom can be defined independently of l. We will assume that d_l is even, so for every state (ls), there is another one related by time reversal symmetry $(l\bar{s})$. For simplicity, we specialize to the case $d_l = 2, s = \uparrow, \downarrow$. Let $\hat{c}^\dagger_{ls}$ represent the fermionic creation operator for the state (ls). Using the Anderson pseudo-spin operators [39] (quadratic pairing operators), satisfying the $su(2)$ algebra

$$\left[t_i^3, t_j^\pm\right] = \pm\delta_{ij} t_j^\pm, \qquad \left[t_i^+, t_j^-\right] = 2\delta_{ij} t_j^3, \tag{1}$$

the Richardson pairing Hamiltonian is given by

$$H_P = \sum_{l\in\Lambda} 2\epsilon_l t_l^3 - g\sum_{l,l'} t_l^+ t_{l'}^- = \sum_{l\in\Lambda} 2\epsilon_l t_l^3 - g\mathbf{t}^+\cdot\mathbf{t}^-, \tag{2}$$

where $\mathbf{t} = \sum_l \mathbf{t}_l$ is the total spin operator. It maps to the reduced BCS model

$$\hat{H} = \sum_{\mathbf{p},\sigma} \epsilon_{\mathbf{p}} \hat{c}^\dagger_{\mathbf{p},\sigma}\hat{c}_{\mathbf{p},\sigma} - g\sum_{\mathbf{p},\mathbf{k}} \hat{c}^\dagger_{\mathbf{p}\uparrow}\hat{c}^\dagger_{-\mathbf{p}\downarrow}\hat{c}_{-\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\uparrow} \tag{3}$$

by replacing the translational degrees of freedom by rotational ones, where $l \in \Lambda = \{1, \ldots, n\}$ enumerates the one-particle orbital degrees of freedom, while $s = \uparrow, \downarrow$ indicates the two internal spin states per orbital ($d_l = 2$). The pairing Hamiltonian can be decomposed into the linear combination

$$H_P = 2\sum_{l\in\Lambda} \epsilon_l R_l + g\left[\left(\sum_{l\in\Lambda} t_l^3\right)^2 - \frac{1}{4}\sum_{l\in\Lambda}\left(d_l^2 - 1\right)\right]. \tag{4}$$

At a fixed value of the component t^3 of the total angular momentum, the last term becomes a constant and is dropped from the Hamiltonian. The operators R_l (generalized Gaudin magnets [7]) are given by

$$R_l = t_l^3 - \frac{g}{2}\sum_{l'\neq l} \frac{\mathbf{t}_l \cdot \mathbf{t}_{l'}}{\epsilon_l - \epsilon_{l'}}. \tag{5}$$

These operators solve the Richardson pairing Hamiltonian because [18] they are independent, commute with each other and span all the degrees of freedom of the system. Richardson showed [1, 2] that the exact N-pair wavefunction of his Hamiltonian is given by application of operators $b_k^\dagger = \sum_l \frac{t_l^\dagger}{2\epsilon_l - e_k}$ to vacuum (zero pairs state). The unnormalized N-pair wavefunction reads $\Psi_R(\epsilon_i) = \prod_{k=1}^N b_k^\dagger |0\rangle$. The eigenvalues e_k satisfy the self-consistent algebraic equations

$$\frac{1}{g} = \sum_{p\neq k} \frac{2}{e_k - e_p} + \sum_l \frac{1}{2\epsilon_l - e_k}, \tag{6}$$

which can be given a 2D electrostatic interpretation [21] with energy

$$U(\epsilon_l, e_k) = \frac{2}{g}\left[\sum_{k=1}^N \mathcal{R}e(e_k) - \sum_{l=1}^n \mathcal{R}e(\epsilon_l)\right] \tag{7}$$

$$+ 2\sum_{l=1}^n \sum_{k=1}^N \log|e_k - 2\epsilon_l| - 4\sum_{k<p} \log|e_k - e_p| - \sum_{i<j} \log|2\epsilon_i - 2\epsilon_j|. \tag{8}$$

Equations (6) appear as equilibrium conditions for a set of charges of strength $q = 2$ placed at points e_k, in the presence of fixed charges of strength $q = -1$ at points $2\epsilon_l$, and uniform electric field of strength $\frac{1}{g}$, pointing along the real axis. This interpretation proves to be very useful for the conformal field theory (CFT) description of the Richardson problem. The electrostatic energy (7) is minimized for values $\{e_k\}$ corresponding to pair energies. In (7), n, N represent the number of single-particle levels and the number of pair energies, respectively. For a large interaction constant g, the equilibrium positions $\{e_k\}$ form a set of complex conjugated pairs defining a curve γ in the complex plane of energies. We note that the eigenvalues r_i of Gaudin Hamiltonia are proportional in this language to the values of the electric field at positions $2\epsilon_i$, $2r_i = g\frac{\partial U}{\partial \epsilon_i}$.

For a set of single-particle energies $\{\epsilon_i\}$, the BCS ground state is obtained by minimizing the electrostatic energy (7) with respect to positions of the free charges at $\{e_k\}$. Once found, they also determine exactly the values of the electric field at positions $\{2\epsilon_i\}$ on the real axis, which are proportional to the ground-state eigenvalues $\{r_i^{GS}\}$. For any other values of $\{r_i\}$, the electrostatic energy (7) is not minimized. This indicates that for arbitrary values $r_i \neq r_i^{GS}$, the system is not in equilibrium.

2.2. The mean field limit of Richardson–Gaudin models

General description of the classical model. In the mean-field limit, the spin operators $\mathbf{t}_l$ are replaced by their quantum-mechanical averages. Written in terms of the classical vectors $\mathbf{S}_l = 2\langle \mathbf{t}_l \rangle$, the semiclassical approximation for the pairing Hamiltonian becomes

$$H_{MF} = \sum_{l\in\Lambda} \epsilon_l S_l^3 - \frac{g}{4}|J^-|^2, \tag{9}$$

where $\mathbf{J} = \sum_{l\in\Lambda} \mathbf{S}_l$ and the BCS gap function is given by $\Delta = gJ^-/2$. Replacing commutators by canonical Poisson brackets,

$$\left\{S_i^\alpha, S_j^\beta\right\} = 2\epsilon^{\alpha\beta\gamma} S_i^\gamma \delta_{ij}, \tag{10}$$

variables S_i^α become smooth functions of time. In this limit, the problem can be analysed with tools of classical integrable systems, and the solution is known to be exact as $n \to \infty$.

The Poisson brackets (10) and Hamiltonian (9) lead to the equations of motion

$$\dot{\vec{S}}_i = 2(-\vec{\Delta} + \epsilon_i \hat{z}) \times \vec{S}_i, \tag{11}$$

where $2\vec{\Delta} = (gJ_x, gJ_y, 0)$ and $\vec{J}$ is the total spin. The semiclassical limits of the Gaudin Hamiltonian are independent constants of motion,

$$r_i = \frac{1}{2}\left[S_i^z - \frac{g}{2}\sum_{j\neq i} \frac{\vec{S}_i \cdot \vec{S}_j}{\epsilon_i - \epsilon_j}\right], \qquad \dot{r}_i = 0. \tag{12}$$

Equations (11) describe a set of strongly interacting spins and have generic nonlinear oscillatory solutions. The exact solution may be obtained through the Abel–Jacobi inverse map [9, 15, 45].

This solution can be described exactly in the language of hyperelliptic Riemann surfaces (see the appendix for details). At this point, it is useful to make use of the intuitively clear features of this construction (figure 1). For a given set of initial conditions for the spins $\{\vec{S}_i\}$, i.e. also of the constants of motion $\{r_i\}$, a polynomial $Q(u)$ of degree $2n$ and with n pairs of complex conjugated roots $E_{2k+2} = \overline{E}_{2k+1}$, $k = 0, \ldots, n-1$, is constructed. A schematic representation of these roots is given in figure 1. Between each pair of roots, we place a simple cut $\mathcal{C}_k = [E_{2k+1}, E_{2k+2}]$ on the complex plane of energies. The surface thus obtained is a representation of a torus of smooth genus $g = n - 1$.

Variables u_k are introduced for $n-1$ of these cuts, with respect to which the equations of motion separate. The variables u_k evolve in time in a complicated fashion, solving a system of nonlinear coupled differential equations (A.4). Up to a constant, the time dependence of the gap parameter amplitude is given by

$$\log|\Delta(t)| = \mathcal{I}m \int u(t)\,\mathrm{d}t, \qquad u(t) = \sum_k u_k(t). \tag{13}$$

The widths of the cuts $\mathcal{C}_k$ and the periods of the nonlinear oscillators u_k are determined by the values of constants of motion $\{r_i\}$. For the particular choice $r_i = r_i^{GS}$, all the cuts $\mathcal{C}_k$, $k = 1, \ldots, n-1$ vanish, and the width of the remaining cut $\mathcal{C}_n$ equals the equilibrium value of the gap function: $\left|E_{2n-1}^{GS} - E_{2n}^{GS}\right| = 2|\Delta|^{GS}$. In that case, the oscillators $u_k = E_{2k-1} = E_{2k}$ are at rest, and the only time dependence left in the system is the uniform precession of the parallel planar spins S_i^-, with frequency $\omega = 2\sum_{k=1}^n \epsilon_k - 2\sum_{p=1}^{n-1} E_{2p-1} - \sum_{i=1}^n S_i^z$. In the case of particle–hole symmetry, ω vanishes as well.

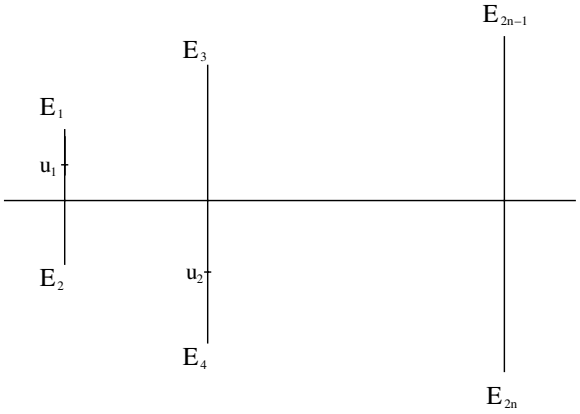

Figure 1. Schematic representation of the Liouville torus for the integrable system (11).

2.3. Perturbations and phase averaging

The effect of random, uncorrelated perturbations of the single-particle spectrum on the nonlinear dynamics of Anderson pseudo-spins is complicated at the smallest time scale of the system, consisting essentially in random dephasing of the collective motion. However, if we are no longer interested in the dynamics at time scale τ_Δ, but rather in the long time asymptotics at $t = O(\tau_\delta)$, the analysis simplifies considerably. The fast time scale may be eliminated entirely by phase averaging over the cycles of the unperturbed solution, which is justified in the case of uncorrelated noise. Clearly, details of the dynamics at τ_Δ are lost by this procedure; however, the result is a universal behaviour on the slow scale, which depends only on the global topological classification of the unperturbed solution, or the unperturbed spectral curve. Mathematically, the procedure is not trivial, and deserves a separate discussion. We provide these details in the next section.

3. Phase averaging for hyperelliptic spectral curves

We consider the unperturbed problem given by a nonlinear scalar differential equation $R(u, \dot{u}, \ddot{u}, \ldots) = 0$, where R is a rational function of the solution $u(t)$ and its derivatives $\dot{u}, \ddot{u}, \ldots$. Specializing to the cases described by hyperelliptic spectral curves, we express it through the Lax pair of 2×2 operators L, A, solving the linear vector problem

$$L\Psi = \mu\Psi, \qquad \partial_t \Psi = A\Psi, \qquad \partial_t L = [A, L], \tag{14}$$

where λ, μ are two auxiliary complex variables, $A(\lambda, u)$ is a 2×2 matrix and $\Psi(\lambda, \mu, t)$ is a column vector. The operators are chosen such that elements of the identity $\dot{L} = [A, L]$ are equivalent to $R(u, \dot{u}, \ddot{u}, \ldots) = 0$. A trivial calculation gives the isospectral property $\dot{\mu} = 0$.

Specializing the operator L to have the form [46] (the u functional dependence is implicit throughout)

$$L(\lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & -a(\lambda) \end{bmatrix}, \tag{15}$$

the eigenvalue equation for Ψ becomes

$$\mu^2 + \det L = 0, \qquad \mu(\lambda) = \pm \mathrm{i}\sqrt{\det L(\lambda)}. \tag{16}$$

When functions a, b, c in (15) are rational, equation (16) defines the hyperelliptic Riemann surface $\mu(\lambda)$ called *spectral curve* of the system (14). As noted before, the time evolution leaves the spectral curve invariant, which is why the problem is sometimes called *isospectral*.

The isomonodromic deformation [44] is introduced through

$$L\Psi = \mu\Psi + \epsilon\partial_\lambda\Psi, \qquad [\epsilon\partial_\lambda - L, \partial_t - A] = 0, \tag{17}$$

where the second equation is the isomonodromy requirement, and $0 \leqslant \epsilon \ll 1$. We recover the unperturbed problem by setting $\epsilon = 0$. The first equation is easily integrated and gives the formal solutions

$$\log\Psi_\pm(\lambda, \mu, t) = \frac{1}{\epsilon}\left[-\mu\lambda \pm \mathrm{i}\int^\lambda \sqrt{\det L(\sigma)}\,\mathrm{d}\sigma\right], \tag{18}$$

up to constants in λ. Let us now impose the saddle-point (or turning-point) condition $\partial_\lambda \log\Psi = 0$. This will give $\mu = \pm\mathrm{i}\sqrt{\det L}$, i.e. the spectral curve. Therefore, we may see the isospectral problem as a saddle-point (turning point) approximation. The compatibility equations

$$[\epsilon\partial_\lambda - L, \partial_t - A] = 0 \tag{19}$$

can be recast in the form

$$\epsilon(\partial_\nu L - \partial_\lambda A) + \partial_\tau L - [A, L] = 0, \tag{20}$$

where we have split the time dependence ∂_t into a fast time scale ∂_τ and a slow one $\epsilon\partial_\nu$. At zero order in ϵ, (20) is simply the unperturbed problem. The first-order correction gives the slow-time scale dependence of the modulated solution $u_\epsilon(\tau, \nu)$. Moreover, from the matrix elements of (20), we get after averaging over the fast motion in τ [47],

$$\partial_\nu \det L = -(\overline{2aA_{11} + bA_{21} + cA_{12}}), \tag{21}$$

where the bar signifies τ-averaging. Since for the unperturbed problem, $\partial_\tau \det L = 0$, the averaging is justified.

Equation (21) is simply the result of Bogoliubov–Whitham [48] averaging for the problem (17). It tells us how the previously invariant spectral curve now changes slowly in time over the large time scale ν. It also gives us the form of the deformed solution $u_\epsilon(\tau, k_i(\nu))$, as a *modulation* of the original solution.

Note. The fact that such perturbations generalize the autonomous Garnier system of [15] to a non-autonomous system of Schlesinger type was indicated in [47]. Extensions of these systems to include a constant matrix were discussed in [49].

4. Phase averaging and the nonequilibrium BCS problem

4.1. Topological classification of finite-gap solutions

In the presence of spectrum symmetry $\epsilon_k = -\epsilon_{-k}$, $S_k^z = -S_{-k}^z$, the distribution of cuts $\mathcal{C}_k$ obeys the same symmetry. The number of non-degenerate cuts is therefore even. This analysis uses the fact that at $t = 0$ all Anderson pseudo-spins are aligned along the z axis. In the simplest non-trivial case, there are only two non-degenerate cuts as shown in figure 2 and $g = 1$, while the corresponding variable u_1 is given by elliptic functions. Other non-trivial cuts $\mathcal{C}_k$ may exist in general, associated with dynamics of variables u_k. In the limit of 'small' cuts [50], their separate contributions are given by trigonometric functions and the behaviour of the gap parameter takes the simplified form

$$\log\frac{|\Delta(t)|}{|\Delta(0)|} = \mathcal{I}m\int u(t)\,\mathrm{d}t = \mathcal{U}_{\mathrm{ell}}(t) + \sum_k \mathcal{U}^k_{\mathrm{trig}}. \tag{22}$$

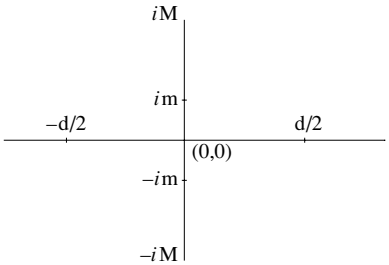

Figure 2. Distribution of complex roots E_i around the origin, for the elliptic approximation. The mean level spacing is d.

As we shall see, the most relevant contribution is due to the elliptic part $\mathcal{U}_{\rm ell}$, studied in the next section.

4.2. Modulations of elliptic solutions

For the root distribution shown in figure 2, there is only one variable u_k, taking imaginary values [15]. It solves an equation of the type [15]

$$(\dot{u})^2 + (u^2 + m^2)(u^2 + M^2) = 0. \tag{23}$$

For simplicity, we therefore make the transformation $u \to \mathrm{i}u$, keeping the time variable real. The new function is a real quartic oscillator which satisfies

$$(\dot{u})^2 = (u^2 - M^2)(u^2 - m^2). \tag{24}$$

The solutions corresponding to this distribution of roots are

$$u_1(t) = m \cdot \mathrm{sn}(Mt + \phi_1, k_1), \qquad k_1 = m/M, \tag{25}$$

$$u_2(t) = M \cdot \mathrm{sn}(mt + \phi_2, k_2), \qquad k_2 = M/m, \tag{26}$$

where sn is the Jacobi sine function, and $\phi_{1,2}$ are arbitrary phases. In the degenerate case $m = M$, the solutions become hyperbolic functions. Solution u_2 is non-physical in our case.

In order to set up the isomonodromic deformation method, consider the Lax pair [51]:

$$L = -[2\dot{u}\sigma_1 + 4u\lambda\sigma_2 + (4\lambda^2 - \xi + 2u^2)\mathrm{i}\sigma_3], \tag{27}$$

$$A = \mathrm{i}\lambda\sigma_3 - u\sigma_2, \qquad \epsilon\partial_\lambda\Psi = L\Psi, \qquad \partial_t\Psi = A\Psi, \tag{28}$$

where σ_α, $\alpha = 1, 2, 3$, are the Pauli matrices, ϵ is a small real number and $\Psi = (\psi_1, \psi_2)^t$ is the Baker–Akhiezer function. Choosing the initial condition $\psi_1 = 0$, $\psi_2 = 1$, we can extract the amplitude of the ratio $\Delta(t)/\Delta(0)$ from $\Psi(t)$ as

$$\left|\frac{\Delta(t)}{\Delta(0)}\right| = \frac{[\psi_1(t) + \overline{\psi}_1(t)]}{2}. \tag{29}$$

The compatibility (zero-curvature) conditions

$$[\partial_t - A, \epsilon\partial_\lambda - L] = 0 \tag{30}$$

yield the system of equations

$$\partial_t\xi = \epsilon, \qquad \partial_t^2 u = 2u^3 - \xi u. \tag{31}$$

We note that

$$\det L = -4\{(\dot{u})^2 - [u^4 - \xi u^2 + (2\lambda^2 - \xi/2)^2]\}. \tag{32}$$

Setting $\epsilon = 0$ gives the equation $(\dot{u})^2 - u^4 + \xi u^2 =$ constant, while the limit $\epsilon = 1$ yields the Painlevé II equation. In fact, the unperturbed case $\epsilon = 0$ allows us to retrieve the full elliptic solution, from the equation

$$L\Psi = 0, \qquad \det(L) = 0, \tag{33}$$

which gives the elliptical function u satisfying

$$(\dot{u})^2 - [u^4 - \xi u^2 + (2\lambda^2 - \xi/2)^2] = 0. \tag{34}$$

The physical solution u_1 is obtained identifying

$$\xi = m^2 + M^2, \qquad 2\lambda = M + m. \tag{35}$$

Now restore $\epsilon \neq 0$, write $\partial_t = \partial_\tau + \epsilon\partial_\xi$ and retain terms of order ϵ from (30). Averaging over the fast variable τ gives

$$\partial_\xi \det L = \overline{L_{22}\partial_\lambda A_{11}} + \overline{L_{11}\partial_\lambda A_{22}}, \tag{36}$$

or equivalently,

$$\partial_\xi \det L = -(4\overline{u^2} - 2\xi + 8\lambda^2). \tag{37}$$

Performing the computations, we obtain

$$\partial_\xi[u^4 - \xi u^2 - (\dot{u})^2] = -\overline{u^2}. \tag{38}$$

Writing the elliptic equation as

$$(\dot{u})^2 = u^4 - \xi u^2 + \mu^2, \tag{39}$$

the physical solution takes the form

$$u_1(t) = \sqrt{\frac{\xi - \sqrt{\xi^2 - 4\mu^2}}{2}}\,\mathrm{sn}\left[\sqrt{\frac{\xi + \sqrt{\xi^2 - 4\mu^2}}{2}}t\right],$$

up to an arbitrary initial phase ϕ, and elliptic modulus

$$k^2 = \frac{1 - \sqrt{1 - \left(\frac{2\mu}{\xi}\right)^2}}{1 + \sqrt{1 - \left(\frac{2\mu}{\xi}\right)^2}}. \tag{40}$$

The Whitham averaging equation has the form

$$4\frac{\partial\mu^2}{\partial\xi^2} = 1 - \frac{2}{2 - k^2}\frac{\mathcal{E}(k^2)}{\mathcal{K}(k^2)}, \tag{41}$$

where $\mathcal{E}, \mathcal{K}$ are the complete elliptic integrals of the first and second kinds, respectively. Equation (41) has a fixed point at $k = 0$, $\frac{\mu}{\xi} \to 0$. This shows that, on the slow time scale, the parameter $\mu/\xi = m/M + O(m^3/M^3)$ goes to zero, as $\xi = m^2 + M^2$ increases. Expanding the solution in this limit, and integrating under the separation of time scales assumption, we obtain for the elliptic contribution to the gap parameter, the approximation

$$\Delta_{\mathrm{ell}}(t) = \Delta(0)\,\mathrm{e}^{k\int \mathrm{sn}(\tau,k^2)\mathrm{d}\tau}, \tag{42}$$

and $k^2 = \frac{\mu^2}{\xi^2} + O(\mu^4/\xi^4) \to 0$ as $t \to \infty$.

Asymptotic behaviour of modulated elliptic solutions. Starting from the Lax pair (27, 28), we can obtain the asymptotic behaviour of $\Psi(\lambda, t)$ as $t \to \infty$, in the whole complex λ plane [51].

The analysis is simpler when working in the variable $z = \frac{\lambda}{\sqrt{\xi}}$. There are six Stokes sectors at $z \to \infty$, with canonical asymptotes for Ψ, but the region of interest is $z = 1/2$, where

$$\frac{4\lambda^2}{\xi} \to 1, \qquad \frac{\mu^2}{\xi^2} \to 0. \tag{43}$$

Using the Whitham equation, separation of scales, and identifying the strength of the fluctuations ϵ with an effective temperature T, we obtain for the elliptic contribution at large times (22)

$$U_{\rm ell}(t) = \mathcal{F}(t)\frac{\tau_\delta}{t}, \tag{44}$$

where $\mathcal{F}$ is a bounded function $|\mathcal{F}| = O(1)$.

Modulations of trigonometric solutions. In order to analyse the slow dynamics of the small cuts, let $\xi \to \infty$ in (39) and write the solution as

$$u = \frac{\mu}{\sqrt{\xi}}\cos(\sqrt{\xi}t + \phi), \qquad \mu^2/\xi \to 0. \tag{45}$$

This assumption is consistent with (39). Formally, this is simply the trigonometric approximation of the general elliptic solution, in the limit of small modulus k^2. However, the parameter ξ in (45) must be sent to ∞ much faster than the physical $\xi = M^2 + m^2$, in order to give the correct spherical limit. The contributions from the small cuts therefore vanish faster than the elliptic component.

Higher-genus contributions. In the case where there are several non-degenerate cuts at $t = 0$ corresponding to a high genus $g > 1$, the isomodromic deformation method can be applied in the same way as for $g = 1$, leading to modulated hyper-elliptic solutions of Painlevé equations degree higher than 2. There are few systematic results in this field, and a complete classification does not exist at this time. Specific high-genus solutions may have interesting topological properties with physical interpretations in terms of the collective excitations of the Gaudin magnet.

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Appendix. Isospectral case and the Abel–Jacobi inversion problem

In [9, 15], the system (11) with a fixed spectral curve was solved through inverse Abel–Jacobi mapping, by using Sklyanin separation of variables techniques [8, 52]. Interesting connections to generalized Neumann systems and Hitchin systems were discovered in [24]. The solution starts from the Lax operator

$$\mathcal{L}(\lambda) = \frac{2}{g}\sigma_3 + \sum_{i=0}^{n}\frac{\vec{S}_i\cdot\vec{\sigma}}{\lambda - \epsilon_i} = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & -a(\lambda)\end{bmatrix}, \tag{A.1}$$

where $\sigma_\alpha, \alpha = 1, 2, 3$, are the Pauli matrices, and λ is an additional complex variable, the spectral parameter. Let $u_k, k = 1, \ldots, n-1$, be the roots of the coefficient $c(\lambda)$. Poisson brackets for variables S_i^α read

$$\left\{S_j^\alpha, S_k^\beta\right\} = 2\epsilon_{\alpha\beta\gamma}S_k^\gamma\delta_{jk}. \tag{A.2}$$

The Lax operator (A.1) defines a Riemann surface (the spectral curve) $\Gamma(y,\lambda)$ of genus $g = n-1$, through

$$y^2 = Q(\lambda) = \det \mathcal{L}(\lambda) \left[g \frac{P(\lambda)}{2} \right]^2, \tag{A.3}$$

where $P(\lambda) = \prod_{i=1}^{n} (\lambda - \epsilon_i)$.

The equations of motion for the Hamiltonian (9) become

$$\dot{u}_i = \frac{2iy(u_i)}{\prod_{j\neq i}(u_i - u_j)}, \qquad i\dot{J}^- = J^- \left[gJ^3 + 2\sum_{k=1}^{n} \epsilon_k - u \right]. \tag{A.4}$$

In (A.4), $u = -2\sum_{i=1}^{n-1} u_i$, $b(u_i) = 0$.

From the equations of motion, it is clear that knowledge of the initial amplitude of J^- and of the roots $\{u_i\}$ is enough to specify the n unit vectors $\{\mathbf{S}_i\}$, for a given set of constants of motion $\{R_l\}$ given by the classical limit of Gaudin Hamiltonia. The Dubrovin equations (A.4) are solved by the inverse of the Abel–Jacobi map, as we explain in the following. We begin by noting that the polynomial $Q(\lambda)$ has degree $2n$, and is positively defined on the real λ-axis. Therefore, the curve $\Gamma(y,\lambda)$ has n cuts between the pairs of complex roots $[E_{2i-1}, E_{2i}]$, $i = 1, 2, \ldots, n$, perpendicular to the real λ-axis. The points u_i belong to $n-1$ of these cuts, $u_i \in [E_{2i-1}, E_{2i}]$, $i = 1, \ldots, n-1$. These $g = n-1$ cuts allow us to define a canonical homology basis of Γ, consisting of cycles $\{\alpha_i, \beta_i\}$, $i = 1, \ldots, g$. With respect to these cycles, a basis of normalized holomorphic differentials $\{\omega_i\}$ can be defined, through

$$\mu_i = \lambda^{g-i} \frac{\mathrm{d}\lambda}{y}, \qquad M_{ij} = \int_{\alpha_j} \mu_i, \qquad \boldsymbol{\omega} = M^{-1}\boldsymbol{\mu}. \tag{A.5}$$

The period matrix $B_{ij} = \int_{\beta_j} \omega_i$ is symmetric and has a positively defined imaginary part. The Riemann θ function is defined with the help of the period matrix as

$$\theta(\boldsymbol{z}|B) = \sum_{\boldsymbol{n}\in\boldsymbol{z}^g} \mathrm{e}^{2\pi \mathrm{i}(\boldsymbol{n}^t \boldsymbol{z} + \frac{1}{2}\boldsymbol{n}^t B \boldsymbol{n})}. \tag{A.6}$$

The g vectors $\boldsymbol{B}_k$ consisting of columns of B and the basic vectors $\boldsymbol{e}_k$ define a lattice in $\mathbb{C}^g$. The *Jacobian* variety of the curve Γ is then the g-dimensional torus defined as the quotien $J(\Gamma) = \mathbb{C}^g/(\mathbb{Z}^g + B\mathbb{Z}^g)$. The Abel–Jacobi map associates with any point P on Γ, a point (g- dimensional complex vector) on the Jacobian variety, through $\boldsymbol{A}(P) = \int_{\infty}^{P} \boldsymbol{\omega}$. Considering now a g-dimensional complex vector of points $\{P_k\}$, $k = 1, \ldots, g$, on Γ, defined up to a permutation, we can associate with it the point on the Jacobian

$$\boldsymbol{z} = \boldsymbol{a}(\boldsymbol{P}) = \sum_{k=1}^{g} \boldsymbol{A}(P_k) + \boldsymbol{K}, \tag{A.7}$$

where $\boldsymbol{K}$ is the Riemann characteristic vector for Γ.

The map (A.7) suggests that we now have a way to describe the dynamics on Γ by following the image point on the Jacobian. Given a point on the g-dimensional Jacobian $\boldsymbol{z} = (\zeta_1, \ldots, \zeta_{n-1})$, we can find a unique set of points $\{\lambda_k\}$, $k = 1, \ldots, g$ on Γ, such that $\boldsymbol{z} = \boldsymbol{a}(\boldsymbol{\lambda})$, and $\theta(\boldsymbol{a}(\boldsymbol{P}) - \boldsymbol{z}|B) = 0$. The system evolves in time according to the point $\boldsymbol{z}(t)$

$$\zeta_k = ic_k, \qquad 1 \leqslant i \leqslant g-1, \qquad \zeta_{n-1} = i(c_{n-1} + t), \tag{A.8}$$

where $\{c_k\}$ is a set of initial conditions, such that $\boldsymbol{z}_0 = \boldsymbol{z}(t=0) = \boldsymbol{a}(\boldsymbol{c})$, and $\boldsymbol{c}$ is the set of initial conditions for positions of $\boldsymbol{\lambda}$ on Γ. Together with the initial condition which determines the initial amplitude of J^-, this set will determine entirely the evolution of the functions $u_i(t)$, $J^-(t)$.

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